Reflection Monoids

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Motivation

To know what does Reflection monoid mean?

Let us review

- What is S_n?

Long time ago: We know the Symmetric group as

Permutations of $[n] = \{1, 2, 3, \ldots n\}$.

We can also think about it in other version

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Linear isomorphisms of V permuting the basis vectors $\{v_1, v_2, \ldots, v_n\}$

For all $\sigma \in S_n$, define $g_\sigma \in GL(V)$ by:

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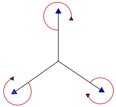
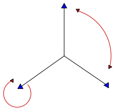
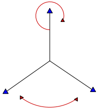
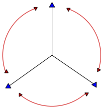
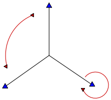
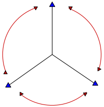
The symmetric group as linear isomorphisms

Example: if $n = 3$, then S_3 permuting $\mathbb{R}^3$, as the following

$\pi \in S_3$	g_π on the basis of $\mathbb{R}^3$	$\pi \in S_3$	g_π on the basis of $\mathbb{R}^3$
id		(23)	
(12)		(123)	
(13)		(132)	

What about I_n

Question:

- Can we think of the symmetric inverse monoid I_n in term of linear maps?

Well! Let us remember what is I_n?

- $I_n = \{\text{bijections } Y \to Z : Y,\ Z \subseteq [n]\}$

Undefined elsewher   Partial identiy   Zero partial permutation

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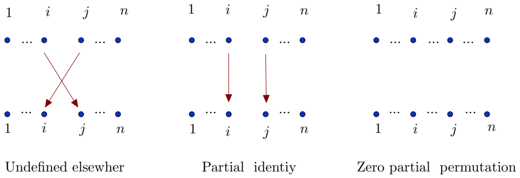


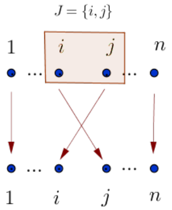
The other way to describe I_n

The other equivalent way to describe elements of I_n is as follows:

- Take a permutation $\pi \in S_n$ and restricted to some subsets $J \subseteq [n]$. Hence, we will get a partial permutation π_J.

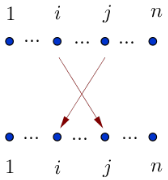
The other way to describe I_n

$\pi =$



Restricting a permutation $\pi \in S_n$ to a subset J gives partial permutation π_J

$\pi_J =$



$i \rightarrow j$

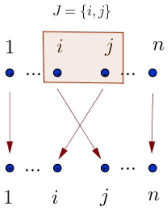
$j \rightarrow i$

undefined elsewhere

π_J is not a unique expression

The other way to describe I_n

$\pi =$



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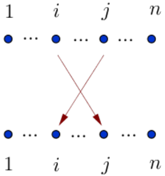

$i \to j$

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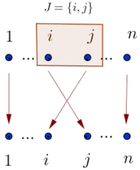
The other way to describe I_n

$$\pi_J = \sigma_I \iff J = I \text{ and the permutation } \pi\sigma^{-1} \text{ fixes } J \text{ pointwise}$$

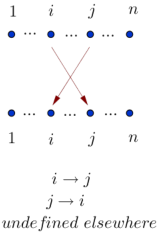
The other way to describe I_n

Well! we can think of a linear version of I_n, as follows:

$\pi =$



π *(can be seen as linear isomprphism)* g_π



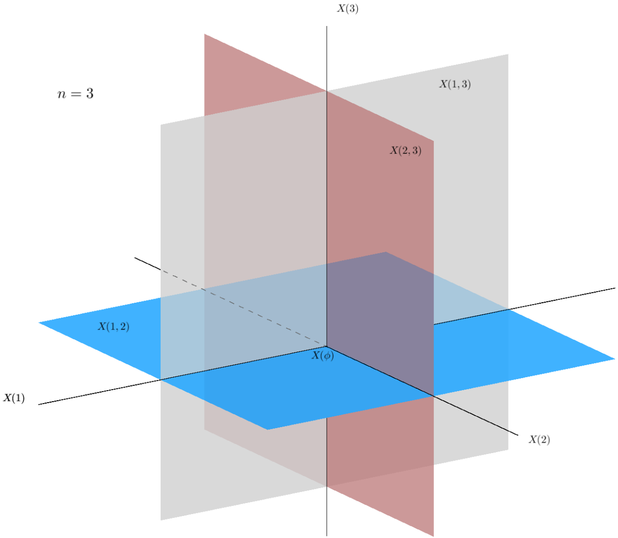
$= \pi_J \rightsquigarrow g_\pi|_{X(J)}$

where $g_\pi : V \to V$

and $X(J)$ *subspace of* V

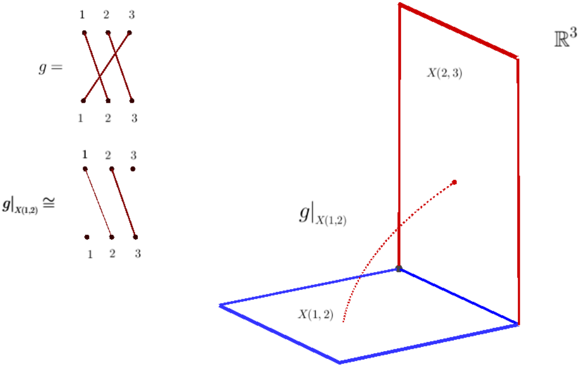
What about I_n

The collection of subspaces of $\mathbb{R}^3$:



Restriction S_n to subspaces

Take $g \in S_3$ and restrict it to a hyperplane spanned by (v_1, v_2), we get



Restriction S_n to subspaces

So far: we have learned the following

- Viewing I_n in terms of partial linear isomorphisms
- Partial permutations $\equiv$ Full permutation restricted to subsets,
- Partial linear isomorphisms $\equiv$ Full linear isomorphisms restricted to subspaces.

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Restriction a subgrouop of $GL(V)$ to subspaces

This draws our attention to define the general linear monoids:

- Let V be a finite vector space
- Define the following monoid
  $ML(V) = \{\text{Partial linear isomorphisms } Y \to Y', \text{ for } Y, Y' \subseteq V\}$
- $\alpha : Y \to Y'$, and $\beta : Z \to Z'$,
- We know $\alpha = g_{_Y}$ and $\beta = h_{_Z}$, where $g, h \in GL(V)$, and $Y = \text{dom}(\alpha)$, $Z = \text{dom}(\beta)$
- $\alpha\beta = g_{_Y} h_{_Z} = (gh)_{_{Y \cap Zg^{-1}}}$

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A system in a vector space

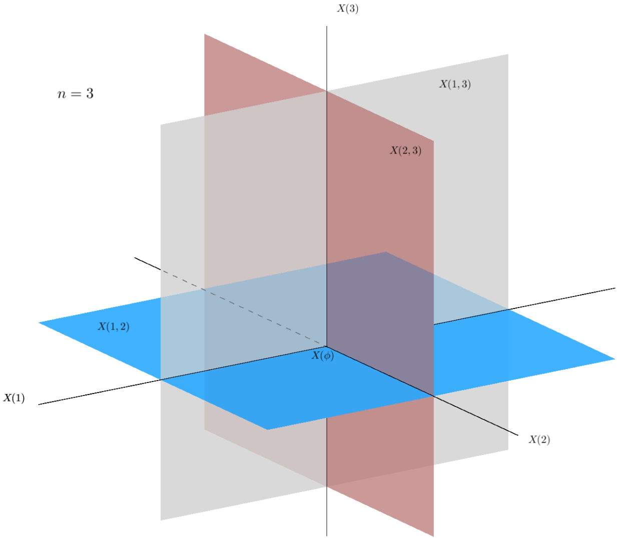
This allow us to define the following:

Definition

Let V be a vector space and $G \leq GL(V)$. A set S of subspaces of V is called a system in V for G if and only if

- $V \in S$,
- $Y \in S$, $g \in G$, $Yg \in S$, and
- $Y, Z \in S$, $Y \cap Z \in S$.

The collection of subspaces of $\mathbb{R}^3$:



Reflection monoids

Definition

Let G be a subgroup of $GL(V)$ and S be a system in V for G. A submonoid of $ML(V)$ defined by

$$M(G,S) := \{g_Y : g \in G, Y \in S\}$$

is called a monoid of partial linear isomorphisms given by a group G and a system S.

Recall g_Y is a partial linear isomorphism $Y \mapsto Yg$ defined by:

$$(v)g_Y = \begin{cases} vg & v \in Y, \\ \text{undefined} & v \notin Y. \end{cases}$$

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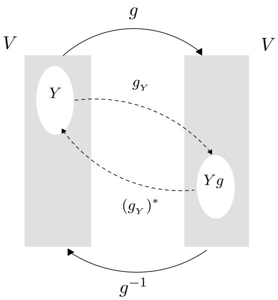

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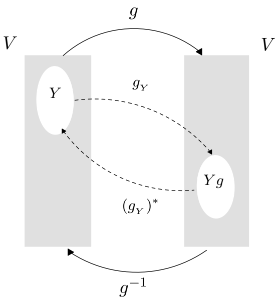

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reflection monoid (version 1) $M(W, S)$

reflection group → W

System → S

Reflection monoids

It turns out that a reflection monoid $M(W,S)$ is a factorizable inverse monoid.

Definition

An inverse monoid M is said to be factorizable, if $\forall\ x \in M,\ \exists$ a unit $g \in U$ and an idempotent $e \in E$ s.t $x = eg$; that is, $M = EU$.

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Coxeter arrangment monoids

Example:

- V be a Euclidean vector space
- $W \subset GL(V)$ be a finite reflection group
- $\mathcal{A} = \{H : \text{ for each reflection } s_H \in W\}$.
- $\mathcal{H} = V \ \cup \ \{ \bigcap_i H_i : \ H_i \in \mathcal{A}\}$ is a system for W.

Hence $M(W, \mathcal{H})$ is a reflection monoid

Definition

A partial reflection is a partial linear isomorphism s_Y, where s is a reflection and Y is a subspace of V.

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Another characterisation of reflection monoids

It turns out

- A factorizable inverse submonoid $M \subset ML(V)$ generated by partial reflections is also a reflection monoid

reflection monoid (version 2) $:= \langle \text{partial reflections } s_Y \rangle \subset ML(V)$

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Orders of reflection monoids

Theorem: Let

- G be a finite subgroup of $GL(V)$
- S be a finite system in V, then

$$|M(G,S)| = \sum_{Y \in S} [G : G_Y],$$

where G_Y is the isotropy group of $Y \in S$.

$$G_Y = \{g \in GL(V) : vg = v \ , \ \forall \ v \in Y\}.$$

The order of the Boolean monoids

Recall the Boolean monoids:

- V be a Euclidean vector space with basis $\{v_1, v_2, \ldots, v_n\}$
- $S_n \subset GL(V)$ act on V by permuting the coordinates,
- Fix $L \subseteq I = \{1, 2, ..., n\}$ $X(L) = \bigoplus_{l \in L} \mathbb{R} v_l$ a subspace of V
- The Boolean system $\mathcal{B} = \{X(L) : L \subseteq I\}$

The order of the Boolean monoid $M(S_n, \mathcal{B})$ is obtained by

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